Asymptotically Optimal Spherical Codes[1]

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Abstract — **A new class of spherical codes is presented which are designed analogously to laminated lattice construction. For many minimum angular separations, these "laminated spherical codes" outperform the best known spherical codes. In fact, for fixed dimension $k \leq 49$, the density of the laminated spherical code approaches the density of the $(k-1)$-dimensional laminated lattice Λ_{k-1}, as the minimum angular separation $\theta \to 0$. In particular, the three-dimensional laminated spherical code is asymptotically optimal, in the sense that its density approaches the Fejes Tóth upper bound as $\theta \to 0$. The laminated spherical codes are also structured, which simplifies decoding.**

A *spherical code* $\mathcal{C}(k,\theta)$ is a set of points on the surface of a k-dimensional unit radius sphere S_k having minimum angular separation θ. The *density* of $\mathcal{C}(k,\theta)$, denoted $\Delta_{\mathcal{C}(k,\theta)}$, is the ratio of the surface area of $|\mathcal{C}(k,\theta)|$ disjoint spherical caps centered at the codepoints and with angular radius $\theta/2$, to the surface area of S_k. Let $\Delta(k,\theta) = \max_{\mathcal{C}(k,\theta)} \Delta_{\mathcal{C}(k,\theta)}$. Note that the maximum number of codepoints in any k-dimensional spherical code with minimum angular separation θ can be determined directly from $\Delta(k,\theta)$. We refer to a family of codes $\mathcal{C}(k,\theta)$ as *asymptotically optimal* if $\Delta_{\mathcal{C}(k,\theta)}/\Delta(k,\theta) \to 1$ as $\theta \to 0$.

For fixed dimension k and small minimum angular separation θ, [Fej59] ($k=3$) and [Cox68] ($k \geq 4$) provide the tightest upper bound and [GHSW87] provides the tightest known lower bound on $\Delta(k,\theta)$. However, there is a gap between these bounds as $\theta \to 0$. In this paper we introduce a new spherical code construction analogous to laminated lattice construction. We call these codes *laminated spherical codes.* These new codes have larger asymptotic (for small θ) densities than any previously known spherical codes.

The laminated spherical codes are obtained by placing codepoints on concentric $(k-1)$-dimensional spheres and projecting each codepoint onto S_k by adding a kth coordinate to form a vector of unit norm. The set of points on each $(k-1)$-dimensional sphere is either a $(k-1)$-dimensional laminated spherical code, or another code formed from its deep holes. By nesting the concentric spheres closely, and placing codepoints of one sphere at the radial extension of the deep holes of codepoints of the previous sphere, a method similar to constructing laminated lattices (e.g., [CS93]) is used to construct our spherical codes, which we denote by $\mathcal{C}^\Lambda$. As more of these concentric spheres are stacked up, codepoints start spreading out, and the density lessens. To counteract this, a buffer zone is placed between concentric spheres, and a new, tighter packed $(k-1)$-dimensional spherical code is placed in the next sphere. A recursion describes the sequence of radii necessary to insure that both the desired minimum angular separation is maintained and the desired density is obtained.

Our construction has similarities to those of [Yag58] and [GHSW87] in that a projection from $k-1$ dimensions to k dimensions is used; the difference lies in the placement of points prior to the projection. Our technique is practical for creating codes of any size and thus provides a lower bound on achievable minimum distance as a function of code size.

Let $\Delta_{\mathcal{C}^\Lambda}(k) = \limsup_{\theta\to 0} \Delta_{\mathcal{C}^\Lambda(k,\theta)}$, and let Δ_{Λ_k} be the density of the sphere packing constructed from the laminated lattice Λ_k. In the laminated spherical code, layers ($(k-1)$-dimensional spheres) are stacked similarly to layers of lattices in a laminated lattice, and as a result, $\Delta_{\mathcal{C}^\Lambda}(k)$ is equal to the density of the sphere packing generated by Λ_{k-1}.

Theorem 1 $\Delta_{\mathcal{C}^\Lambda}(k,d) = \Delta_{\Lambda_{k-1}} - O(d^{1/k})$.

Corollary 1 *$\mathcal{C}^\Lambda(3,d)$ is asymptotically optimal and the Fejes Tóth upper bound is asymptotically tight.*

Corollary 2 *If there exists a family of spherical codes $\mathcal{C}(k,d)$ whose asymptotic density is higher than $\Delta_{\mathcal{C}^\Lambda}(k,d)$, then there exists a $(k-1)$-dimensional sphere packing denser than that generated by Λ_{k-1}.*

Theorem 2 *There is an optimal decoder for $\mathcal{C}^\Lambda(k,\theta)$ using $O(\sqrt{|\mathcal{C}^\Lambda(k,\theta)|})$ space and $O(\log|\mathcal{C}^\Lambda(k,\theta)|)$ time, or an optimal decoder using $O(1)$ space and $O(\sqrt{|\mathcal{C}^\Lambda(k,\theta)|})$ time.*

References

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