Capacity of Pulse Position Modulation on Gaussian and Webb Channels

Jon Hamkins

Sam Dolinar

Contents

- Introduction

Contents

- Introduction
  - ⋆ Channel Description

Contents

- Introduction
  - ⋆ Channel Description
  - ⋆ General Capacity Formula

Contents

- Introduction
  - ⋆ Channel Description
  - ⋆ General Capacity Formula
  - ⋆ Capacity Bound and Asymptotic Behavior

Contents

- Introduction
  - ⋆ Channel Description
  - ⋆ General Capacity Formula
  - ⋆ Capacity Bound and Asymptotic Behavior
- Statistical Models of the Optical Channel:

Contents

- Introduction
  - ⋆ Channel Description
  - ⋆ General Capacity Formula
  - ⋆ Capacity Bound and Asymptotic Behavior
- Statistical Models of the Optical Channel: AWGN

Contents

- Introduction
  - ⋆ Channel Description
  - ⋆ General Capacity Formula
  - ⋆ Capacity Bound and Asymptotic Behavior
- Statistical Models of the Optical Channel: AWGN, Webb

Contents

- Introduction
  - ⋆ Channel Description
  - ⋆ General Capacity Formula
  - ⋆ Capacity Bound and Asymptotic Behavior
- Statistical Models of the Optical Channel: AWGN, Webb, Webb+Gaussian

Contents

- Introduction
  - ⋆ Channel Description
  - ⋆ General Capacity Formula
  - ⋆ Capacity Bound and Asymptotic Behavior
- Statistical Models of the Optical Channel: AWGN, Webb, Webb+Gaussian
- Capacity as a Function of:

Contents

- Introduction
  - ⋆ Channel Description
  - ⋆ General Capacity Formula
  - ⋆ Capacity Bound and Asymptotic Behavior
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- Capacity as a Function of:
  - ⋆ Fundamental Parameters

Contents

- Introduction
  - ⋆ Channel Description
  - ⋆ General Capacity Formula
  - ⋆ Capacity Bound and Asymptotic Behavior
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  - ⋆ Fundamental Parameters
  - ⋆ Bit-SNR

Contents

- Introduction
  - ⋆ Channel Description
  - ⋆ General Capacity Formula
  - ⋆ Capacity Bound and Asymptotic Behavior
- Statistical Models of the Optical Channel: AWGN, Webb, Webb+Gaussian
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  - ⋆ Fundamental Parameters
  - ⋆ Bit-SNR
  - ⋆ Physical Parameters

Contents

- Introduction
  - ⋆ Channel Description
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  - ⋆ Fundamental Parameters
  - ⋆ Bit-SNR
  - ⋆ Physical Parameters
- Hard vs. Soft Output Channel

Contents

- Introduction
  - ⋆ Channel Description
  - ⋆ General Capacity Formula
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  - ⋆ Fundamental Parameters
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Contents

- Introduction
  - ⋆ Channel Description
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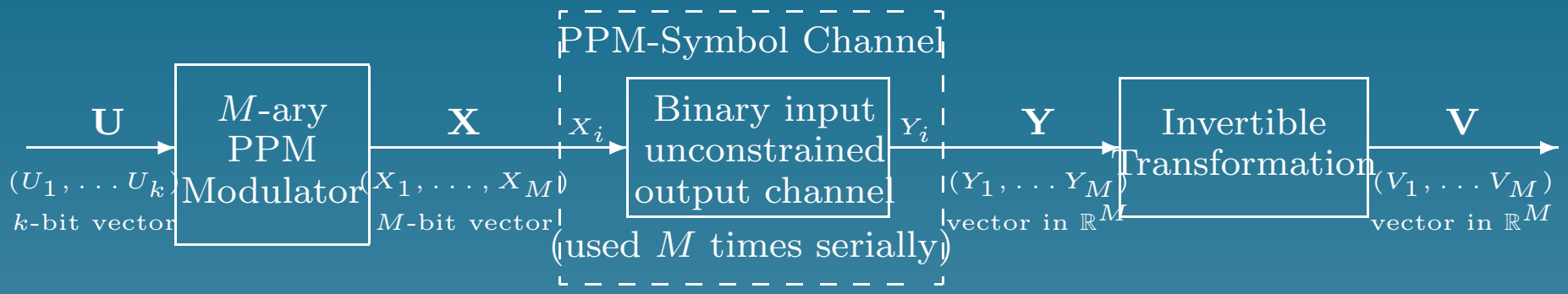
  - ⋆ Bit-SNR
  - ⋆ Physical Parameters
- Hard vs. Soft Output Channel
- Capacity Sensitivity
- Conclusions

Channel Description



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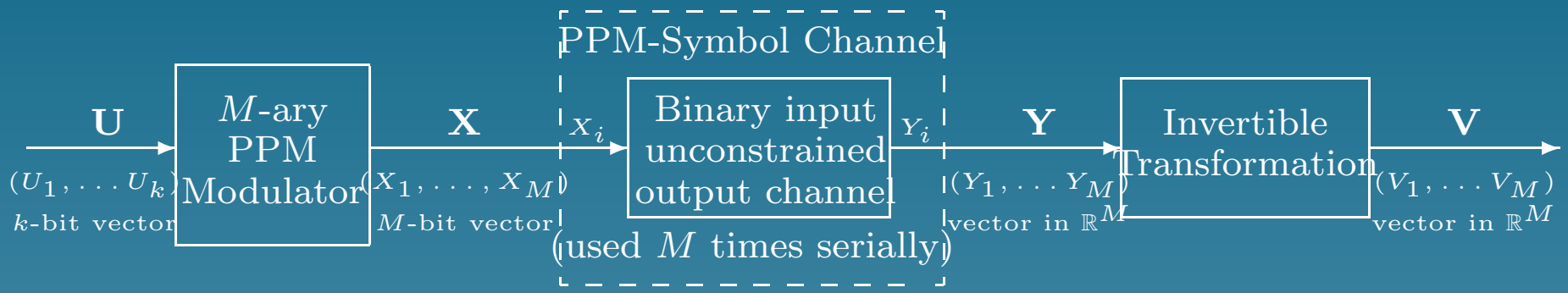
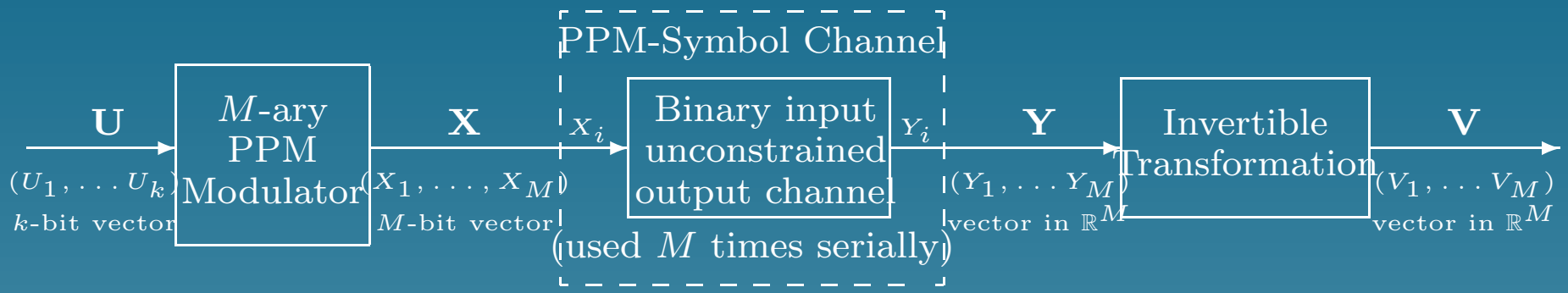


Signal Flow:

Channel Description



Signal Flow:

- k-bit block of information

  Example: $\mathbf{U} = (U_1, \ldots, U_4) = (0, 1, 0, 1)$

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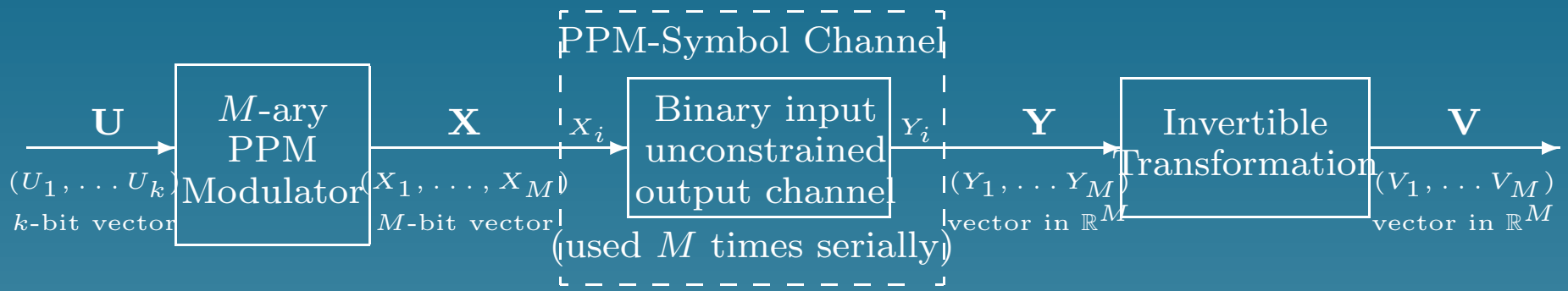

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- Modulated to one of $M = 2^k$ PPM symbols $\{\mathbf{s}_1, \ldots, \mathbf{s}_M\}$

  Example: $\mathbf{X} = (X_1, \ldots, X_{16}) = \mathbf{s}_5 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0)$

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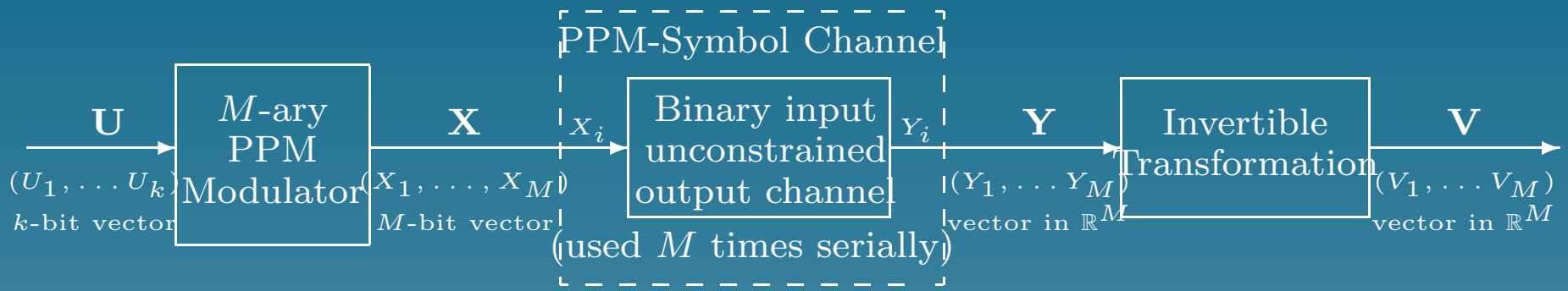

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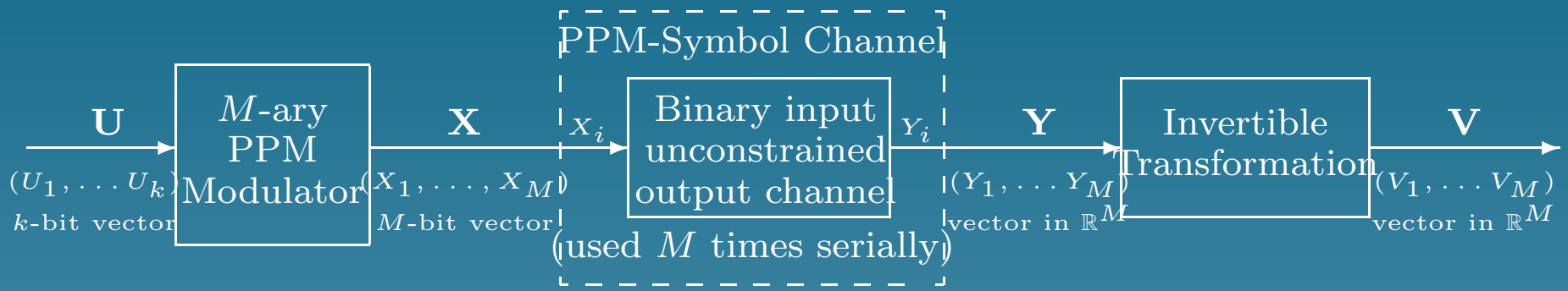
- Each bit of symbol is transmitted on binary-input memoryless channel

  Example: $\mathbf{Y} = (Y_1, \ldots, Y_{16}) =$

  $(0.1, -0.6, 0.0, 0.3, 1.1, 0.6, -0.4, 1.0, 0.5, 0.5, 11.7, 0.4, -0.2, 0.0, 0.1, -0.7)$

Channel Description



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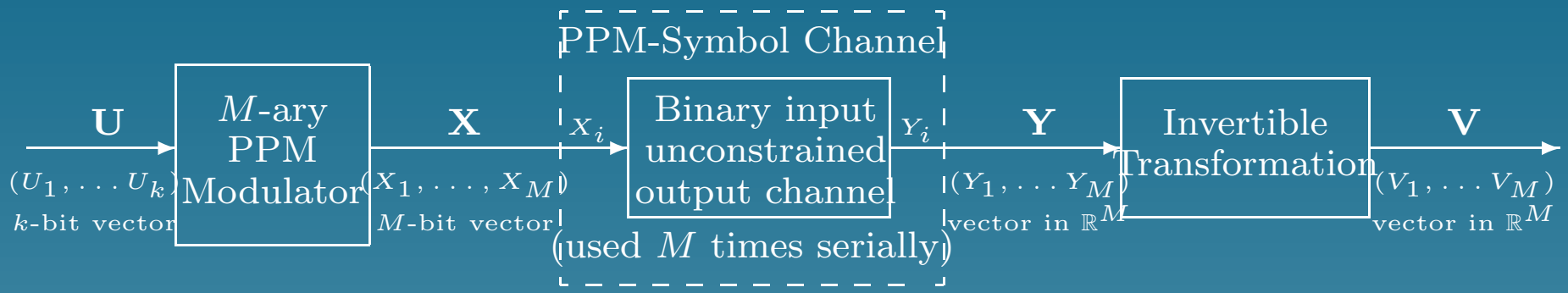
- A componentwise deterministic invertible function is applied

  Example: $V_j = 2(U_j - 0.1) \Rightarrow \mathbf{V} = (V_1, \ldots, V_{16}) =$

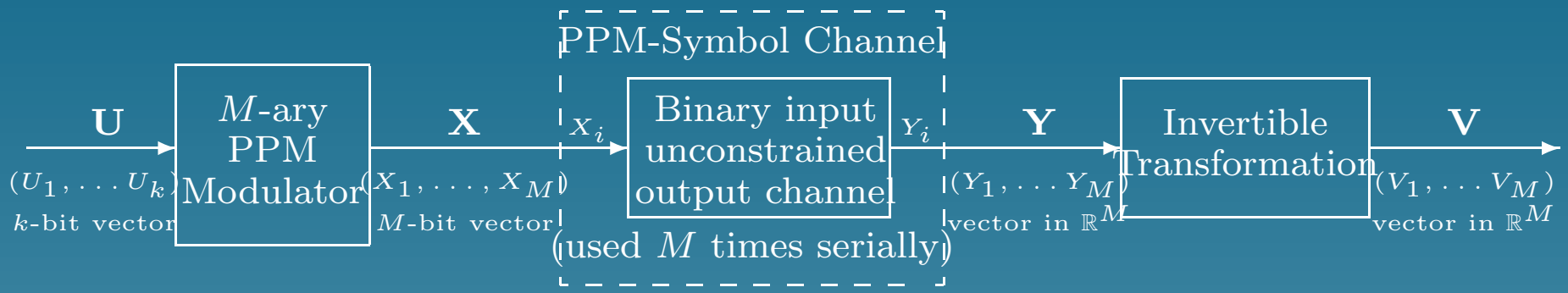
  $(0.0, -1.4, -0.2, 0.4, 2.0, 1.0, -1.0, 1.8, 0.8, 0.8, 23.2, 0.6, -0.6, -0.2, 0.0, -1.2)$

Channel Description



Statistical notation:

Channel Description



Statistical notation:

$p_0(\cdot)$: The conditional pdf of V_i given $X_i = 0$

$p_1(\cdot)$: The conditional pdf of V_i given $X_i = 1$

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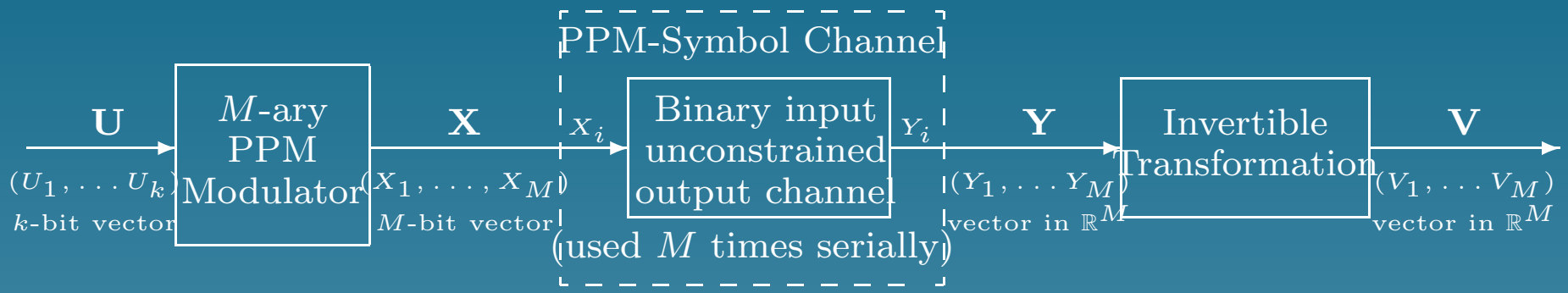

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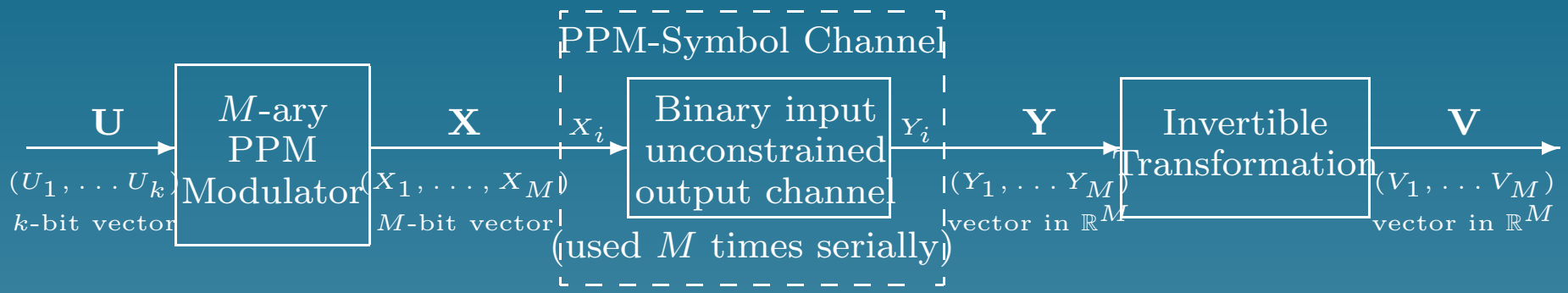
The elements of $\mathbf{V}$ are independent, and consequently,

$$p(\mathbf{V} = \mathbf{v}|\mathbf{X} = \mathbf{s}_j) = \prod_{i=1}^{M} p(V_i = v_i|\mathbf{X} = \mathbf{s}_j) = p_1(v_j) \prod_{i=1, i\neq j}^{M} p_0(v_i),$$

Capacity



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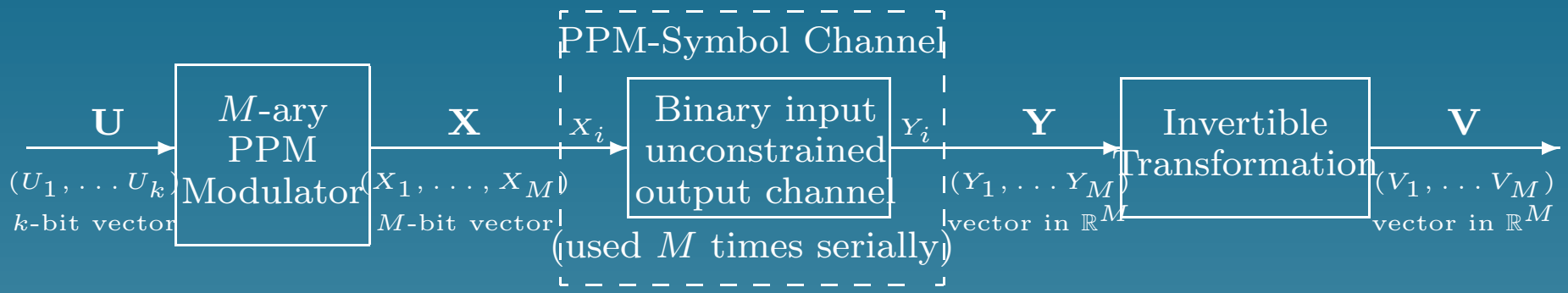
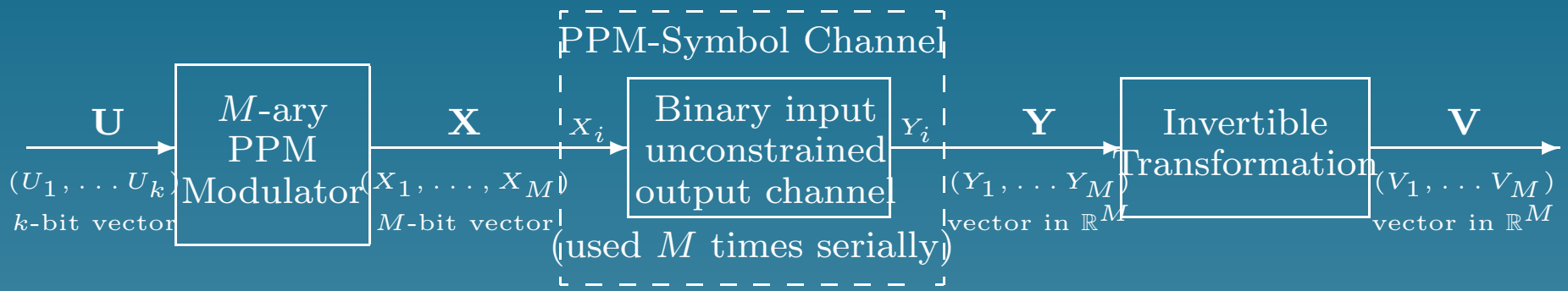


Capacity:

$$C \triangleq \max_{p(\mathbf{U})} I(\mathbf{V};\mathbf{U}) \text{ bits per channel use}$$

Capacity



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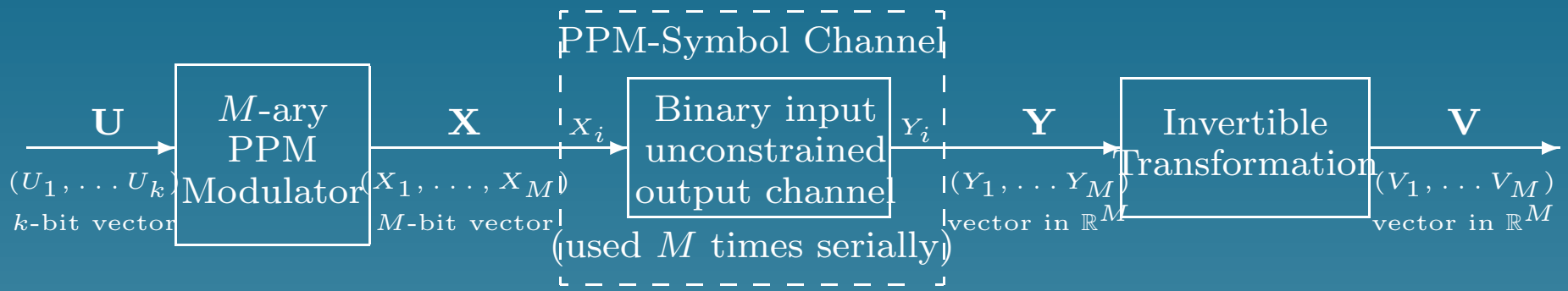
$$= \max_{p(\mathbf{X})} I(\mathbf{V};\mathbf{X})$$

Capacity



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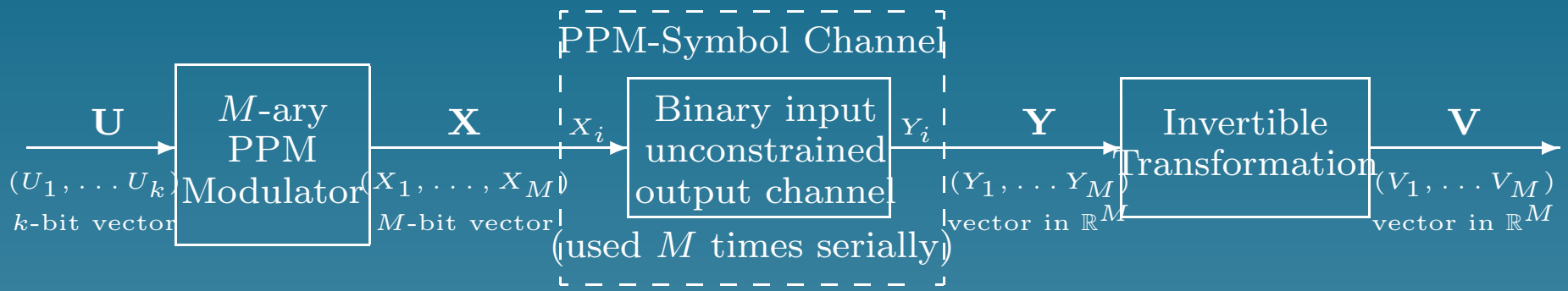
$$
\begin{aligned}
C &\triangleq \max_{p(\mathbf{U})} I(\mathbf{V};\mathbf{U}) \text{ bits per channel use} \\
&= \max_{p(\mathbf{X})} I(\mathbf{V};\mathbf{X}) \\
&= \max_{p(\mathbf{X})} \sum_{i=1}^{M} \int_{\mathbb{R}^M} p(\mathbf{X}=\mathbf{s}_i) p(\mathbf{V}=\mathbf{v}|\mathbf{X}=\mathbf{s}_i) \log_2 \left[\frac{p(\mathbf{V}=\mathbf{v}|\mathbf{X}=\mathbf{s_i})}{p(\mathbf{V}=\mathbf{v})} \right] d\mathbf{v}
\end{aligned}
$$

Capacity



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&= \int_{\mathbb{R}^M} p(\mathbf{V}=\mathbf{v}|\mathbf{X}=\mathbf{s}_1) \log_2 \left[\frac{p(\mathbf{V}=\mathbf{v}|\mathbf{X}=\mathbf{s_1})}{p(\mathbf{V}=\mathbf{v})} \right] d\mathbf{v}
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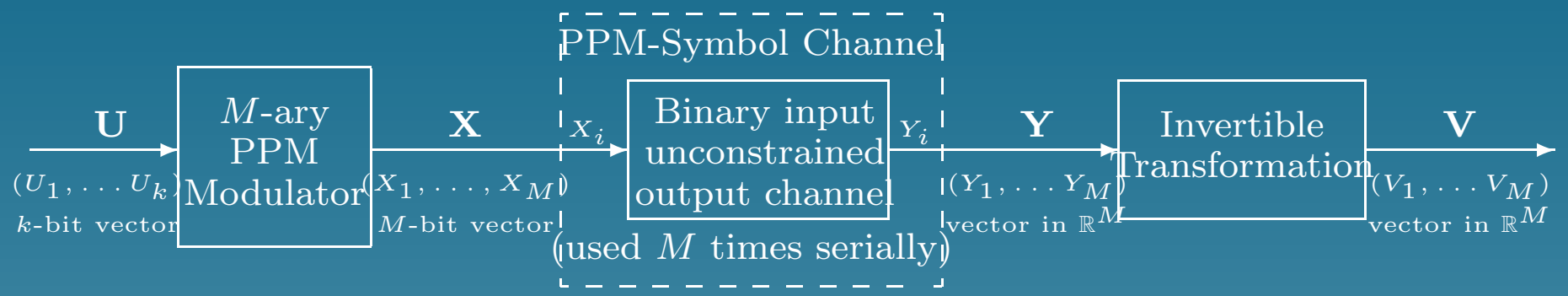
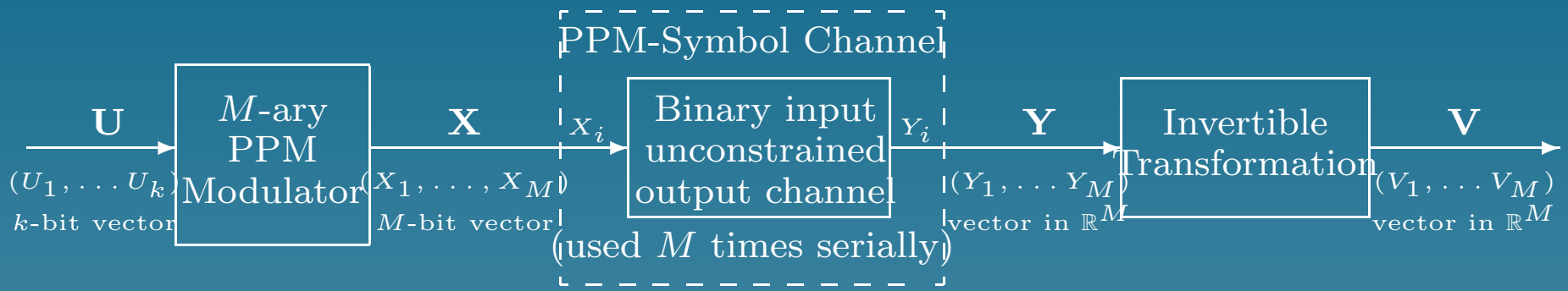


Capacity:

$$C = \int_{\mathbb{R}^M} p(\mathbf{V} = \mathbf{v} | \mathbf{X} = \mathbf{s}_1) \log_2 \left[\frac{p(\mathbf{V} = \mathbf{v} | \mathbf{X} = \mathbf{s_1})}{\frac{1}{M} \sum_{j=1}^{M} p(\mathbf{V} = \mathbf{v} | \mathbf{X} = \mathbf{s}_j)} \right] d\mathbf{v}$$

Capacity



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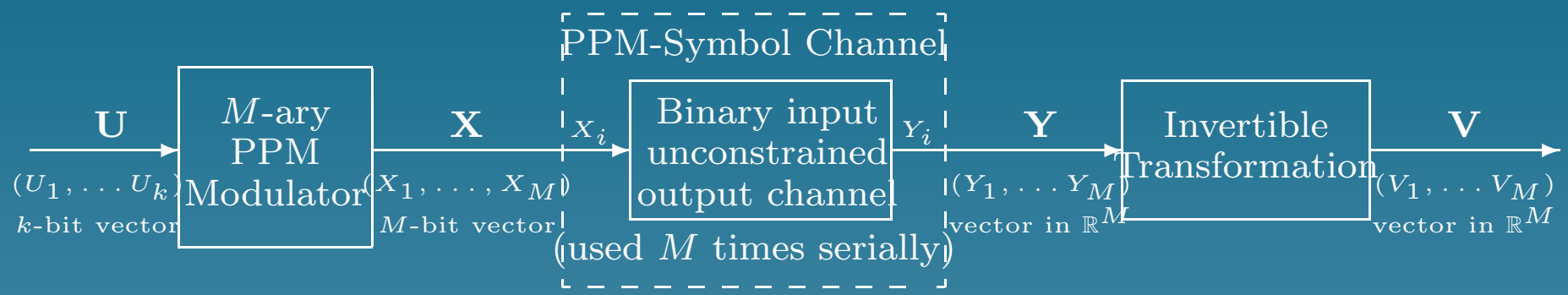
&= \int_{\mathbb{R}^M} p(\mathbf{V}=\mathbf{v}|\mathbf{X}=\mathbf{s}_1) \log_2 \left[\frac{M p_1(v_1) p_0(v_j)}{\sum_{j=1}^{M} p_0(v_1) p_1(v_j)} \right] d\mathbf{v}
\end{aligned}
$$

Capacity



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&= \int_{\mathbb{R}^M} p(\mathbf{V}=\mathbf{v}|\mathbf{X}=\mathbf{s}_1) \log_2 \left[\frac{M p_1(v_1) p_0(v_j)}{\sum_{j=1}^{M} p_0(v_1) p_1(v_j)} \right] d\mathbf{v} \\
&= E \log_2 \left[\frac{M L(V_1)}{\sum_{j=1}^{M} L(V_j)} \right],
\end{aligned}
$$

where $L(x) \triangleq p_1(x)/p_0(x)$, and where $V_1 \sim p_1(\cdot)$ and $V_j \sim p_0(\cdot)$, $j \geq 2$.

Asymptotic Capacity

The capacity can be rewritten as

$$C = E \log_2 \left[\frac{M L(V_1)}{\sum_{j=1}^{M} L(V_j)} \right]$$

Asymptotic Capacity

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$$C = E \log_2 \left[\frac{M L(V_1)}{\sum_{j=1}^{M} L(V_j)} \right] = E \log_2 \left[\frac{M}{1 + \frac{1}{L(V_1)} \sum_{j=2}^{M} L(V_j)} \right].$$

Asymptotic Capacity

The capacity can be rewritten as

$$C = E\log_2\left[\frac{M\,L(V_1)}{\sum_{j=1}^{M} L(V_j)}\right] = E\log_2\left[\frac{M}{1+\frac{1}{L(V_1)}\sum_{j=2}^{M} L(V_j)}\right].$$

By Jensen's inequality,

$$E\log_2\left[1+\frac{1}{L(V_1)}\sum_{j=2}^{M} L(V_j)\right] \le E\log_2\left[1+\frac{1}{L(V_1)}\sum_{j=2}^{M} E\,L(V_j)\right].$$

Thus, for all M,

$$C \ge E\log_2\left[\frac{M}{1+\frac{M-1}{L(V_1)}}\right] \text{ bits per channel use,}$$

and for large M, the bound is asymptotically tight.